Estimation of Possibility-Probability Distributions

Balapuwaduge Sumudu Udaya Mendis and Tom D. Gedeon

School of Computer Science, The College of Engineering and Computer Science,
The Australian National University, Australia
{sumudu,tom}@cs.anu.edu.au

Abstract. We demonstrate a theory for evaluating the likelihood of a probability by way of possibility distributions. This theory derives from the standard probability distribution theory by using the possibility to define an arbitrary function whose values are bounded by $[0, 1]$ that represents the confidence that one may have in the outcomes. In other words, when in classic probability theory the probability of an event is represented by an integral of the probability mass over this event, in possibility theory the probability of an event is the integral of the probability mass times the confidence function over the whole space. This theory is then extended in order to define a similar notion to probability distributions, namely Possibility-Probability distributions, which represent, as for probabilities, the possibilities of a calculated probability for a given fuzzy event. In this context, we aim to define an estimation method of such a Possibility-Probability distribution in the case of experimental samples and the corresponding distribution.

Keywords: Possibility, Probability, Probability of Fuzzy Events, and Possibility of Probability.

1 Introduction

When only small samples of data and/or unreliable data is available, first order uncertainty calculations based on ordinary probability is questioned in the literature [1,2]. In this paper, we introduce an enhanced theory for evaluating the likelihood of the probability by way of possibility distributions. Basically, this theory derives from the standard probability distribution theory by using possibility as an an extra confidence measure. The confidence is represented as an arbitrary function whose values lie between 0 and 1, and represent the confidence that one may have in the outcomes. In classic probability theory, the probability of an event is represented by a measurable set, and is the integral of the probability measure over this set. However, in possibility theory the probability of an fuzzy event is the integral of the probability measure weighted by the confidence function over the whole space. This confidence is in general equivalent to the membership function of that particular event.

Next, we extend the method in order to define a similar notion to probability distributions, namely Possibility-Probability Distributions (PPD), which

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represent, as for probabilities, the possibilities of a calculated probability for a given fuzzy event. In this context, we aim to define an algorithmic estimation of such a possibility distribution in the case of experimental samples and the corresponding distributions.

The earliest concept of the probability of a fuzzy event was introduced in Zadeh [3]. In [4], Zadeh's approach further improved and argued that the probability of a fuzzy event must be a fuzzy number rather than a crisp value. Later, Huang [5] introduced an approach to calculate PPD based on his information diffusion technique. This technique is especially capable of coping with small data samples. Therefore, Huang [6,7] suggested his method of Possibility-Probability calculation was suitable for risk evaluation. Huang and Gedeon [8] extended this method for the use of fuzzy events to calculate Possibility-Probability distributions. However, they did not consider the use of the proper probability calculation of fuzzy events [3]. We aim here to enhance the approach in [8] by using proper probability calculation of fuzzy events. Also, we introduce a further generalized model compared to both approaches in papers [5] and [8].

2 Possibility of Probability

In this section, we first demonstrate the method in [6] for the calculation of PPD. Huang [6] provides the following definition for a Possibility-Probability distribution.

Definition 1. *Let (Ω, φ, P) be a probability space, and P be probability measure. Let the possibility that the probability of x occurring is p be $\pi_x(p)$:*

$$\Pi_p = \{\pi_x(p) \mid x \in \Omega, p \in P \ \} \tag{1}$$

and is called a possibility-probability distribution.

In definition 1, they consider PPD of non-fuzzy events. They further use the information diffusion method [5] to deal with limited sample data. The mathematical definition of the information distribution is a mapping from a Cartesian product to the unit interval $[0, 1]$ [6,5,7].

Definition 2. *Let $X = \{x_i \mid i = 1, \ldots, n\}$ and $Y = \{y_j \mid j = 1, \ldots, m\}$, then*

$$\mu : X \times Y \to [0, 1] \tag{2}$$

is called an information distribution of X on Y, if $\mu(x, y)$ has the following properties:

(a) $\forall x \in X$, if $\exists y \in Y$, such that $x = y$, then $\mu(x, y) = 1$ (i.e. μ is reflexive).
(b) For $x \in X$, $\forall y^{'}, y^{''} \in Y$, if $\mid y_{'} - x \mid \leq \mid y_{''} - x \mid$ then $\mu(y_{'} - x) \leq \mu(y_{''} - x)$ (monotonicity).
(c) $\sum_{j=1}^{m} \mu(x_i, y_j) = 1, i = 1, \ldots, n$ (additive).

μ is called a distribution function of X on Y and can be calculated as follows:

$$\mu(x_i, y_j) = \begin{cases} 1 - \frac{|x_i - y_j|}{\triangle_j} & ; if \ \mid x_i - y_j \mid \leq \triangle_j \\ 0 & ; if \ \mid x_i - y_j \mid > \triangle_j \end{cases} \tag{3}$$
$$= q_{ij}$$

where $\triangle_j = y_{j+1} - y_j$ such that $j = 1, \ldots, m$ and $i = 1, \ldots, n$.

2.1 Interval Set Model

Let $X = \{x_i \mid i = 1, \ldots, n\}$ be a sample, $X \subset \mathbb{R}$, and $Y = \{y_j \mid j = 1, \ldots, m\}$ be a discrete universe of X. Huang [6,7] used the following method to calculate a possibility-probability distribution on intervals,

$$I_j = [y_j - \frac{\triangle_j}{2}, y_j + \frac{\triangle_j}{2}], \ y_j \in Y, \tag{4}$$

with respect to the probability values,

$$p_k = \frac{k}{n}, \ k \in \{1, \ldots, n\} \tag{5}$$

Here k is the number of observations that fall into the interval I_j. From here onwards, $\pi_{I_j}(p_k)$ represents the possibility-probability value of that an event occurs in the interval I_j.

2.2 Joining and Leaving Possibility

Let us take that $x_k \in X$. Next, they assume that the possibility of a probability that includes $x_k \in X$ is based on the distance between center $y_j (\in Y)$ and the data point x_k. If $x_k \in X_{I_j}$[1] (see figure 1) then x_k will leave the interval X_{I_j} with the possibility of,

$$q^-_{kj} = \frac{\mid x_k - y_j \mid}{\triangle_j} = 1 - q_{ij} \tag{6}$$

Similarly, if $x_k \notin X_{I_j}$, then x_k will join the interval X_{I_j} with the possibility of,

$$q^+_{kj} = 1 - \frac{\mid x_k - y_j \mid}{\triangle_j} = q_{ij} \tag{7}$$

The following figure 1 illustrates the situation. Now take that cardinality of X_{I_j} is n_j. Huang [6] writes that the possibility of probability of an event x occurs in I_j is $\frac{n_j}{n}$ is:

$$\pi_{I_j}(\frac{n_j}{n}) = 1 \tag{8}$$

[1] X_{I_j} is called the interior set of interval I_j [6] that contains all elements of interval I_j.

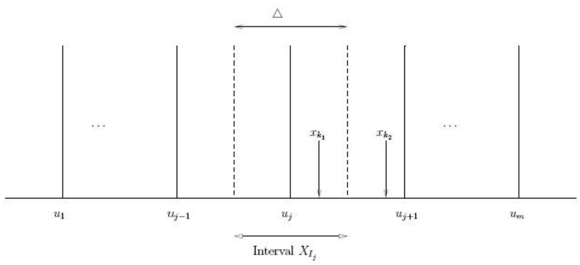

Fig. 1. Intervals on the Universe of Discourse

It is intuitive that $\frac{n_j}{n}$ is the most probable event and thus its possibility is 1. Now, if we assume that $x_k \in X_{I_j}$ may leave the interval I_j when there is a disturbance in the random experiment. Therefore, the possibility that the probability of $x \in I_j$ is $\frac{n_j-1}{n}$ is:

$$\pi_{I_j}(\frac{n_j-1}{n}) = \bigvee_{x_s \in X_{I_j}} q^-_{sj} \tag{9}$$

If two elements leave the interval, we can write that the possibility that the probability of $x \in I_j$ is $\frac{n_j-2}{n}$ is:

$$\pi_{I_j}(\frac{n_j-2}{n}) = \bigvee_{x_{s_1},x_{s_2} \in X_{I_j} | x_{s_1} \neq x_{s_2}} (q^-_{s_1 j} \wedge q^-_{s_2 j}) \tag{10}$$

Similarly, if $x_k \notin X_{I_j}$ may join the interval I_j when there is a disturbance in the random experiment. Therefore, the possibility that the probability of $x \in I_j$ is $\frac{n_j+1}{n}$ is:

$$\pi_{I_j}(\frac{n_j+1}{n}) = \bigvee_{x_s \notin X_{I_j}} q^+_{sj} \tag{11}$$

Finally, when there are n_j observations in interval I_j, we can write a Possibility-Probability distribution of I_j as follows, [6]:

$$\Pi_{I_j}(p) = \begin{cases} \wedge_{x_s \in X_{I_j}} q^-_{sj} & p = p_0 \\ \vdots & \\ \bigvee_{x_{s_1},x_{s_2} \in X_{I_j} | x_{s_1} \neq x_{s_2}} (q^-_{s_1 j} \wedge q^-_{s_2 j}) & p = p_{n_j-2} \\ \bigvee_{x_s \in X_{I_j}} q^-_{sj} & p = p_{n_j-1} \\ 1 & p = p_{n_j} \\ \bigvee_{x_s \notin X_{I_j}} q^+_{sj} & p = p_{n_j+1} \\ \bigvee_{x_{s_1},x_{s_2} \notin X_{I_j} | x_{s_1} \neq x_{s_2}} (q^+_{s_1 j} \wedge q^+_{s_2 j}) & p = p_{n_j+2} \\ \vdots & \\ \wedge_{x_s \notin X_{I_j}} q^+_{sj} & p = p_n \end{cases} \tag{12}$$

Where $p_{n_j} = \frac{n_j}{n}$.

3 Enhanced Possibility-Probability Calculation Methods

In [8] fuzzy intervals are used instead of the mutually exclusive intervals in [6]. The following method of Possibility-Probability calculation can be found in [8].

Let $A = \{A_1, \ldots, A_m\}$ be m fuzzy events in the space of interest. Here $y_j (\in Y)$ in figure 1 is the middle of the core of the fuzzy set A_j in figure 2. Further, figure 2 illustrates all the fuzzyfied intervals. Let us take that $\mid y_j - y_{j+1} \mid = d$ and

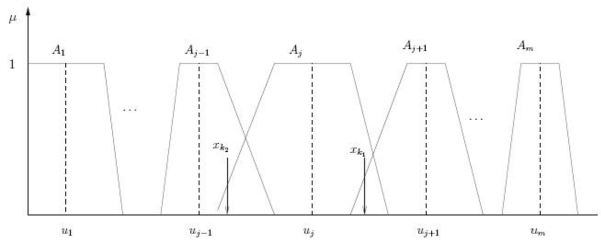

Fig. 2. Intervals on the Universe of Discourse

$X = \{x_1, \ldots, x_n\}$ be the sample space, and these values fall in the interval $[y_j, y_{j+1}]$. Now [8] define S_j and S_{j+1} as follows,

$$S_j = \sum_{i=1}^{n} \mu_{A_j}(x_i) \tag{13}$$

$$S_{j+1} = \sum_{i=1}^{n} \mu_{A_{j+1}}(x_i) \tag{14}$$

$$S = S_j + S_{j+1} \tag{15}$$

Similarly to Huang's method [6], Huang and Gedeon's method [8] also assumes that $\frac{S_j}{S}$ is the maximum possible probability of the event A_j and write

$$\pi_{A_j}(\frac{S_j}{S}) = 1 \tag{16}$$

Now, they consider that the element $x_k \in X$ may leave from fuzzy set A_j, according to the fact that the data point that has the minimum membership to A_j will leave the interval. Therefore, the possibility of probability of A_j occurring being $\frac{S_j - \mu_{A_j}(x_k)}{S}$ as,

$$\pi_{A_j}(\frac{S_j - \mu_{A_j}(x_k)}{S}) = \frac{x_k - y_j}{d} \tag{17}$$

Next if, two points $x_k, x_{k-1} \in X$ may leave the fuzzy set A_j, the the possibility of probability of A_j occurring being $\frac{S_j - \mu_{A_j}(x_k) - \mu_{A_j}(x_{k-1})}{S}$ can be written as

$$\pi_{A_j}(\frac{S_j - \mu_{A_j}(x_k) - \mu_{A_j}(x_{k-1})}{S}) = \frac{x_{k-1} - y_j}{d} \tag{18}$$

where $\mu_{A_j}(x_k) \geq \mu_{A_j}(x_{k-1})$. Conversely, it is possible that some data point will move to the fuzzy set A_j. Suppose that x_t has the minimum membership to A_{j+1} and it is most likely to leave A_{j+1}. Therefore, the possibility of probability of A_j occurring being $\frac{S_j+\mu_{A_{j+1}}(x_t)}{S}$ is,

$$\pi_{A_j}(\frac{S_j + \mu_{A_{j+1}}(x_t)}{S}) = \frac{y_{j+1} - x_t}{d} \tag{19}$$

and the possibility of probability of A_j occurring being $\frac{S_j+\mu_{A_{j+1}}(x_t)+\mu_{A_{j+1}}(x_{+1t})}{S}$ is,

$$\pi_{A_j}(\frac{S_j + \mu_{A_{j+1}}(x_t) + \mu_{A_{j+1}}(x_{t+1})}{S}) = \frac{y_{j+1} - x_{t+1}}{d} \tag{20}$$

where $\mu_{A_{j+1}}(x_t) \geq \mu_{A_{j+1}}(x_{t+1})$. Huang and Gedeon's [8] method can be abstracted into the following form: Let $A = \{A_1, \ldots, A_m\}$ be n fuzzy events in the space of interest. Here $y_j(\in Y)$ is the middle of the core of the fuzzy set A_j. Further, $\mid y_j - y_{j+1} \mid= d$ and $X = \{x_1, \ldots, x_n\}$ be the sample space and the values fall in the interval $[y_j, y_{j+1}]$. Now, let us take that the set $X\prime = \{x_{k+1}, \ldots, x_{k+h} \mid 0 \leq k < n, 0 < h < n\} \subseteq X$, and the possibility of probability of A_j occurring being $\dfrac{S_j - \sum_{t=1}^{h} \mu_{A_j}(x_{k+t})}{S}$ as

$$\pi_{A_j}(\frac{S_j - \sum_{t=1}^{h} \mu_{A_j}(x_{k+t})}{S}) = \frac{\mid y_j - \min_{t=1}^{h} x_{k+t} \mid}{d} \tag{21}$$

The possibility of probability of A_j occurring being $\dfrac{S_j + \sum_{t=1}^{h} \mu_{A_{j+1}}(x_{k+t})}{S}$ is

$$\pi_{A_j}(\frac{S_j + \sum_{t=1}^{h} \mu_{A_{j+1}}(x_{k+t})}{S}) = \frac{\mid y_j - \min_{t=1}^{h} x_{k+t} \mid}{d} \tag{22}$$

4 Possibility-Probability by Employing Probability of Fuzzy Events

Huang's [6,7] method of calculating PPD has serious disadvantages. Firstly, they consider non-fuzzy events to find a possibility of a probability. Therefore, their method can be seen as finding the Possibility-Probability of non-fuzzy events. Secondly, their method assumes that the possibility of an event is proportional to the Euclidian distances between the data points and the centers of those events, and thus lacks a representation of the fuzziness of data against their intervals in the universe.

Huang and Gedeon's [8] measure of the probability of a fuzzy event (eg. probability in equation (16)) is not based on a precise probability measure for a fuzzy event [3]. Further, they consider a special case where all data points in X fall in the interval $[y_j, y_{j+1}]$. In general this not the case, thus equation (15) needs to be improved. In equations (19) and (20), when they write that $\frac{S_j+\mu_{A_{j+1}}(x_t)}{S}$

is the probability of fuzzy set A_i, they violate the concept of the cardinality of a fuzzy set by adding $\mu_{A_{j+1}}(x_t)$ to A_j's cardinality. They also do not consider that A_j's cardinality S_j has already accumulated $\mu_{A_j}(x_t)$. Therefore, they are twice adding x_t's occurrence in the form of $\mu_{A_j}(x_t)$ and $\mu_{A_{j+1}}(x_t)$ to S_j. These are major drawback of Huang and Gedeon's [8] method that we overcome in our methods.

Huang and Gedeon [8] in their first method of finding the possibility of probability also used the same Euclidian distance approach of finding the possibility of their fuzzy events. In their second method, they correctly use fuzzy information, which is the ratio of membership values, to calculate the possibility of probability. However, the second measure is not normalized therefore they need to use an additional scaling function to normalize the data [8].

In this section we provide generalized Possibility-Probability calculation methods that overcome the problems associated with the [6,7] and [8] methods. Additionally, we give a more concrete definition for Possibility-Probability distribution. Unlike Huang's definition in [6], our definition does not consider only a random variable. However, similar to [9], we also use the term "variable", that substitutes for "random variable" in conventional probability theory, considering the fact that a probability measure based on possibility theory can model uncertainty that is caused by more than randomness [1].

Definition 3. *Let $(\mathbb{R}^n, \varphi, P)$ be a probability space, let φ be the σ-field of Borel sets in $\mathbb{R}^n$, and P is a probability measure. Let X be a variable on $\mathbb{R}^n$. Further let A be a fuzzy subset of $\mathbb{R}^n$ in φ. Now, the "Prob(X is A) takes a value in B" induces a possibility distribution $\Pi_{Prob(X\ is\ A)}$ in P:*

$$\Pi(Prob(X\ is\ A)\ = p) = \pi_{Prob(X\ is\ A)} = \mu_B(p) \tag{23}$$

Where B is a fuzzy sub set in P.

Here $\Pi(Prob(X\ is\ A)\ is\ B)$ or is $\Pi_{Prob(X\ is\ A)}$ denoted the PPD of A. Therefore in short we could also denote this as $\Pi(Prob(A))$ or $\Pi_{Prob(A)}$.

Note 1. As we mentioned earlier, in definition 1, Huang considers PPD of non-fuzzy events. In definition 3, we consider PPD of fuzzy events. Thus, the probability "$P(X\ is\ A)$" in the above 2 equations is the probability of the fuzzy event A denoted by [3]:

Definition 4. *When X is discrete,*

$$P(X\ is\ A) = \sum_i \mu_A(x_i) \times P_X(x_i) \tag{24}$$

Here μ_A is the membership function of the fuzzy set A and P_X denotes the probability distribution function of X.

4.1 Approximation of Possibility of Probability: A Generalized Method

Let us take a situation where we have n number of records of data and a permutation $\cdot_{(i)}$ on each $x \in X$ such that $\mu_{A_j}(x_{(1)}) \geq \mu_{A_j}(x_{(2)}) \geq \ldots \geq \mu_{A_j}(x_{(k)}) \geq \mu_{A_j}(x_{(k+1)}) \ldots \geq \mu_{A_j}(x_{(n)})$ where A_j is the j^{th} fuzzy subset on U (in $\mathbb{R}^n$) and X is a variable on U according to definition (3). Also, for the simplicity of the discussion, in the rest of the paper, we write $\mu_{A_j}(x)$ to denote $\mu_{A_j}(u)$ s.t $u = x$ where $u \in U$ and $x \in X$.

Initial Event. Let k be an integer such that at least $k+1 \leq n$ and let $X_k = \{x_{(1)}, \ldots, x_{(k)}\}$ be a sub set of X. Further let us take A_j^k as the k^{th} fuzzy event of the fuzzy set A_j. The membership function of the k^{th} fuzzy event A_j^k can be defined as follows:

$$\mu_{A_j^k(x_i)} = \begin{cases} \mu_{A_j(x_i)} & if\ x_i \in X_k \\ 0 & otherwise \end{cases} \tag{25}$$

Let $n_j^k =\mid X_k \mid= k$ be the cardinality of A_j^k. Now, based on the equation (24), the probability of fuzzy event A_j^k can be calculated as

$$Prob(XisA_j^k) = \sum_{i=1}^{n_j^k} \mu_{A_j^k}(x_i)p(x_i) \tag{26}$$

Now, for the ease of understanding and explaining the leaving and joining possibilities, let us assume that event A_j^k is the starting (initial) event. Similarly to the previous methods the possibility of the initial event is 1.

Leaving Possibility. Next, let the element x_k, which has the lowest membership to the fuzzy event A_j^k, leaves the fuzzy event A_j^k, and let the new resulting fuzzy event state be denoted by A_j^{k-1}. The cardinality of the new fuzzy event A_j^{k-1} is $n_j^{k-1} =\mid X_{k-1} \mid= k-1$. Based on equation (24), the probability of the new fuzzy event, ie. A_j^{k-1}, can be calculated as follows:

$$Prob(XisA_j^{k-1}) = \sum_{i=1}^{n_j^{k-1}} \mu_{A_j^{k-1}}(x_i)p(x_i) = p_{A_j^{k-1}} \tag{27}$$

Next, the possibility that the probability of A_j^{k-1} occurring being $p_{A_j^{k-1}}$ can be calculated as follows

$$\pi(Prob(XisA_j^{k-1})) = 1 - \max_{i=n_j^{k-1}+1}^{n_j^k} \left[\mu_{A_j^k}(x_i)\right] \tag{28}$$

In general, after $l(\leq k)$ data points leave A_j^k, we can calculate the possibility of the fuzzy event A_j^{k-l} occurring being $p_{A_j^{k-l}}$.

$$\pi(Prob(XisA_j^{k-l})) = 1 - \max_{i=n_j^{k-l}+1}^{n_j^k} \left[\mu_{A_j^k}(x_i)\right] \tag{29}$$

This process will continue until all data points will leave the initial fuzzy event A_j^k.

Note 2. Note that in equation (29) when $l = 0$, it gives the possibility that the probability of A_j occurring being $p_{A_j^k}$ as:

$$\pi(Prob(XisA_j^k)) = 1 - \mu_{A_j^k}(x_{k+1}) = 1 - 0 = 1$$

Joining Possibility. Let us assume that the element x_{k+1} will join the fuzzy event A_j^k, and let the new resulting fuzzy event states be A_j^{k+1}. Now the cardinality of the fuzzy event A_j^{k+1} is $n_j^{k+1} =\mid X_{k+1} \mid= k+1$. Now, based on the equation (24), the probability of the new fuzzy event, ie. A_j^{k+1}, can be calculated as follows:

$$Prob(XisA_j^{k+1}) = \sum_{i=1}^{n_j^{k+1}} \mu_{A_j^{k+1}}(x_i)p(x_i) = p_{A_j^{k+1}} \tag{30}$$

Next, the possibility that the probability of A_j^{k+1} occurring being $p_{A_j^{k+1}}$ can be calculated as follows:

$$\pi(Prob(XisA_j^{k+1})) = \min_{i=n_j^k+1}^{n_j^{k+1}} \left[\mu_{A_j^{k+1}}(x_i)\right] \tag{31}$$

In general, after $l(\leq (n-k))$ data points join A_j^k, we can calculate the possibility of the fuzzy event A_j^{k+l} occurring being $p_{A_j^{k+l}}$.

$$\pi(Prob(XisA_j^{k+l})) = \min_{i=n_j^k+1}^{n_j^{k+l}} \left[\mu_{A_j^{k+l}}(x_i)\right] \tag{32}$$

This process will continue until all possible data points will join the initial fuzzy event A_j^k.

Possibility of Probability. In this subsection, we give a generalized method of calculation of the possibility for the all available probabilities for the event A_j.

$$\begin{aligned} \Pi_{Prob(X\ is\ A_j)} &= \sum_{l=0}^{n_j^k} \frac{1 - \max_{i=n_j^{k-l}+1}^{n_j^k} \left[\mu_{A_j^k}(x_i)\right]}{Prob(X\ is\ A_j^{k-l})} \\ &\quad + \sum_{l=n_j^k+1}^{n} \frac{\min_{i=n_j^k+1}^{n_j^{k+l}} \left[\mu_{A_j^{k+l}}(x_i)\right]}{Prob(XisA_j^{k+l})} \end{aligned} \tag{33}$$

Here in equation (33), $\sum$ denote an union operation of a fuzzy number and $P(XisA_j^k)$ is given by equation (26).

5 Conclusion

First we have shown the disadvantages of Huang's [6] method. Secondly, we removed the mathematical irregularities in Huang and Gedeon's method [8]. As a result, we provide a generalized method that estimates a PPD from available data. Importantly, the generalized PPD assume the fuzziness of the events that occurs in reality. Our method can be described as a generalized Possibility-Probability calculation method that uses a possibility theory based approach to estimate the likelihood of the reality of the calculated probability. This method is very useful when only a small sample of data is available.

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